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Quark state confinement as a consequence of the extension of the Bose–Fermi recoupling to $SU(3)$ colour

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Abstract
The Bose–Fermi recoupling of particles arising from the $\mathbb{Z}_2$-grading of the irreducible representations of spin $SU(2)$ is responsible for the Pauli exclusion principle. We demonstrate from fundamental physical assumptions how to extend this to gradings, other than the $\mathbb{Z}_2$ grading, arising from other groups. This requires non-associative recouplings where phase factors arise due to *rebracketing* of states. In particular, we consider recouplings for the $\mathbb{Z}_3$-grading of $SU(3)$ colour and demonstrate that all the recouplings graded by triality leading to the Pauli exclusion principle demand quark state confinement. Note that quark state confinement asserts that only ensembles of triality zero are possible, as distinct from spatial confinement where particles are confined to a small region of space by a *confining force* such as that given by the dynamics of QCD. Finally this result is independent of any algebraic model. One is yet to determine a non-associative field operator algebra realizing such recouplings.

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1. Introduction

Bose–Fermi recoupling leads directly to the Pauli exclusion principle which, for example, underlies the stability of atoms. Observational evidence shows that particles come either as bosons or fermions. Particle statistics arise from the phases associated with the recoupling of states. A recoupling for the representations of $SU(2)$, where a sign change is introduced for interchange of half integer spin and no sign change for interchange involving an integer spin, generates symmetric boson and anti-symmetric fermion states.

In the early days of the quark model it was realized that certain fermionic particle resonances, based on conventional reasoning, appeared to have symmetric states. An example, given in Kaku [1], is the resonance Δ^{++} composed of three up quarks of total spin $\frac{3}{2}$. The state must be symmetric in quark flavour and the spin $\frac{1}{2}$ of each quark must be aligned. The state must also be symmetric in quark spin. Hence the overall state is symmetric, yet the resonance is fermionic. The solution was to introduce $SU(3)$ colour to generate anti-symmetric quark states. Although the existence of quarks is well established, a single free quark has never been observed. We call this quark state confinement. We distinguish this from spatial confinement which accounts for the localization of quarks to a small region of space. The latter arises from the dynamics of a theory such as QCD. We argue that quark state confinement is a result of any $\mathbb{Z}_3$ graded recoupling for $SU(3)$ colour admitting Pauli exclusion of quarks. Furthermore, we determine exactly when a generalized Bose–Fermi grading leads to state confinement.

We consider physical systems conforming to the following assumptions.

(i) The system possesses an exact symmetry given by some semi-simple group G.
(ii) Single particle state spaces are finite unitary irreducible representations of the group G.
(iii) Composite (particle) state spaces are given by coupling together constituent single particle state spaces using tensor product.
(iv) Recoupling of composite state spaces is a natural isomorphism.

The first three assumptions are well-established quantum axioms. The fourth assumption perhaps needs some explanation. A recoupling is an invertible intertwiner (G-equivariant and linear) satisfying a naturality condition. Naturality is an important idea coming from category theory [2]. For example, given three particles with state spaces $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$ in the state ψ_1, ψ_2 and ψ_3 respectively, a recoupling between the physically equivalent state spaces $(\mathcal{H}_1 \otimes \mathcal{H}_2) \otimes \mathcal{H}_3$ and $\mathcal{H}_2 \otimes (\mathcal{H}_1 \otimes \mathcal{H}_3)$ is an invertible intertwiner $T_{\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3} : (\mathcal{H}_1 \otimes \mathcal{H}_2) \otimes \mathcal{H}_3 \to \mathcal{H}_2 \otimes (\mathcal{H}_1 \otimes \mathcal{H}_3)$ recoupling the states $(\psi_1 \otimes \psi_2) \otimes \psi_3$ to something like $\psi_2 \otimes (\psi_1 \otimes \psi_3)$. The natural condition satisfied is that given any individual observation or preparation of the individual states by linear operators $A_i : \mathcal{H}_i \to \mathcal{H}_i'$ changing the state ψ_i to ψ_i', the following diagram commutes:

$$\begin{array}{ccc} (\mathcal{H}_1 \otimes \mathcal{H}_2) \otimes \mathcal{H}_3 & \xrightarrow{T_{\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3}} & \mathcal{H}_2 \otimes (\mathcal{H}_1 \otimes \mathcal{H}_3) \\ \Big\downarrow{\scriptstyle (A_1 \otimes A_2) \otimes A_3} & & \Big\downarrow{\scriptstyle A_2 \otimes (A_1 \otimes A_3)} \\ (\mathcal{H}_1' \otimes \mathcal{H}_2') \otimes \mathcal{H}_3' & \xrightarrow{T_{\mathcal{H}_1',\mathcal{H}_2',\mathcal{H}_3'}} & \mathcal{H}_2' \otimes (\mathcal{H}_1' \otimes \mathcal{H}_3'). \end{array}$$

Normally for $SU(2)$ with Bose–Fermi recoupling the horizontal arrows introduce no phase change, but as we shall see this is not the case for $SU(3)$ colour Bose–Fermi recoupling.

There is a long history of investigation into associative recoupling, beginning with the early work of Green [3]. Green generalized quantization of associative algebras of annihilation and creation operators. Such generalizations led to parastatistics [4, 6, 7, 10], modular statistics [5] and graded Lie algebras [8, 9]. These approaches work with algebras having an associative universal embedding algebra and have been used to describe some features of the quark model. However, this approach has not been able to explain confinement, instead arguing that its origin is dynamical.

In this paper we do not restrict ourselves to associative recoupling. Instead we seek the most general recoupling consistent with the physical requirements of a quantum system exhibiting symmetry. Furthermore, we make no assumptions about the existence of a generalized colour algebra nor attempt to explain the quark model. We simply determine the ramifications of a Bose–Fermi recoupling for $SU(3)$ colour. The non-associativity is required to accommodate Bose–Fermi recouplings over a $\mathbb{Z}_3$-gradation. There is no physical reason why non-associative recouplings are not admissible. In fact the statistical consequence is quark state confinement without taking dynamics into consideration. These results were announced in Joyce [11].

A symmetric monoidal structure of the category of unitary representations provides a framework for describing recoupling, and the Racah–Wigner calculus. We refer the reader elsewhere for an introduction to category theory, group representation theory and the Racah–Wigner calculus. The book by Mac Lane [2] is the standard reference on category theory. An introduction to braided monoidal categories in the context of quantum groups are Kassel [23] and Majid [24]. The group representation notation used in this paper is based on Bröcker and tom Dieck [12]. A gentle introduction to a category theoretic formulation of the Racah–Wigner calculus is given in Joyce *et al* [22] and for coupling theory Joyce [14]. Although category theory is the best language to describe recoupling, we trust that much of the paper is accessżble through examples, and the usage of non-categorical language whenever it is feasible to do so.

We demonstrate in this paper that a Bose–Fermi colour recoupling is neither a symmetric monoidal nor a braided monoidal structure. Colour recoupling requires a symmetric premonoidal structure as defined in Joyce [15, 16]. A symmetric premonoidal structure introduces a natural automorphism to account for the non-commutativity of the pentagon diagram. Hence recouplings based on symmetric premonoidal structures are necessary and lead to a deformation/generalization of the usual Racah–Wigner calculus. This calculus together with appropriate diagram notation is developed in a series of papers [17–21].

2. Recoupling and statistics

The collection of unitary representations for a group G is a symmetric monoidal category URep_G. Loosely it is equipped with a tensor product and recoupling structure. Let Irr_G denote a collection of isotypical irreducible representations (or irreps). Suppose that G is semisimple so that every representation is decomposable as a direct sum of elements from Irr_G. A one-particle ket state of the system is the mapping

$$|\psi\rangle : \mathbb{C} \to (\lambda) \tag{1}$$

given by $z \mapsto z\psi$ where $z \in \mathbb{C}$, $\lambda \in \mathrm{Irr}_G$, the parentheses are the restriction functor Res^G taking $\lambda \mapsto (\lambda) = \mathrm{Res}^G\lambda = \mathbb{C}^{|\lambda|}$ and $\psi \in (\lambda)$. One should think of (λ) as the state space of the particle described by the irrep λ. For example, the spin half irreps state space is two dimensional and spanned by basis vectors corresponding to spin up and spin down along an axis. Multi-particle states are formed by 'tensoring' single particle states together. The irreps, under tensor product, generate the (projected) Racah–Wigner category $\pi\mathrm{RW}_G$. This category inherits the symmetric monoidal structure of URep_G.

Multi-particle states are built out of single particle states, the state space being given by the tensor product of the single particle states. Given n particles contained in n irreps, the state space representing this multi-particle system is dependent on the order and bracketing of irreps. A particular choice is called an ensemble. We abuse notation and call each irrep a particle. The natural isomorphisms of the symmetric monoidal structure reorder and rebracket ensembles. The order in which the n irreps are coupled is represented by a rooted planar binary tree with labelled leaves. This is called a bracketing tree, see Joyce [14]. Operations between bracketing trees are called recouplings.

The recoupling between ensembles is given by the symmetric monoidal structure of $\pi\mathrm{RW}_G$. That is, by associativity ($\mathfrak{a}$), commutativity ($\mathfrak{c}$) and left and right identity ($\mathfrak{l}$ and $\mathfrak{r}$) natural isomorphisms, where we denote the identity irrep by e. These determine respectively, natural isomorphisms

$$\mathfrak{a}_{a,b,c} : (a \otimes b) \otimes c \to a \otimes (b \otimes c) \tag{2}$$

$$\mathfrak{c}_{a,b} : a \otimes b \to b \otimes a \tag{3}$$

$$\mathfrak{l}_a : e \otimes a \to a \tag{4}$$

$$\mathfrak{r}_a : a \otimes e \to a \tag{5}$$

representing rebracketing, adjacent transposition and removal of the vacuum from the left or right. Given any two couplings of a set of irreps, there are a number of differing sequences of the above elementary recouplings transforming one into the other. If these two sequences compose to always give the same natural isomorphism, we say that the structure is coherent. The Mac Lane coherence theorem [2, 25] asserts that a necessary and sufficient condition for coherence is that the pentagon, hexagon and triangle diagrams commute and that commutativity is symmetric. The symmetry of commutativity asserts $\mathfrak{c}_{b,a} = \mathfrak{c}_{a,b}^{-1}$. The pentagon diagram is

$$\begin{array}{ccccc} ((a\otimes b)\otimes c)\otimes d & \xrightarrow{\ _{a\otimes b,c,d}} & (a\otimes b)\otimes(c\otimes d) & \xrightarrow{\ _{a,b,c\otimes d}} & a\otimes(b\otimes(c\otimes d)) \\ \downarrow{\scriptstyle\ _{a,b,c}\otimes 1_d} & & & & \uparrow{\scriptstyle 1_a\otimes\ _{b,c,d}} \\ (a\otimes(b\otimes c))\otimes d & & \xrightarrow[\ _{a,b\otimes c,d}]{} & & a\otimes((b\otimes c)\otimes d). \end{array}$$

The hexagon diagram is

$$\begin{array}{ccccc} (a\otimes b)\otimes c & \xrightarrow{\ _{a,b,c}} & a\otimes(b\otimes c) & \xrightarrow{\ _{a,b\otimes c}} & (b\otimes c)\otimes a \\ \downarrow{\scriptstyle\ _{a,b}\otimes 1_c} & & & & \downarrow{\scriptstyle\ _{b,c,a}} \\ (b\otimes a)\otimes c & \xrightarrow[\ _{b,a,c}]{} & b\otimes(a\otimes c) & \xrightarrow[1_b\otimes\ _{a,c}]{} & b\otimes(c\otimes a). \end{array}$$

Lastly, the triangle diagram is

$$\begin{array}{ccc} (a\otimes e)\otimes b & \xrightarrow{\ _{a,e,b}} & a\otimes(e\otimes b) \\ & {\scriptstyle\ _a\otimes 1_b}\searrow \quad \swarrow{\scriptstyle 1_a\otimes\ _b} & \\ & a\otimes b & \end{array} .$$

We require the states of any composite system to be compatible with the recoupling structure. That is, given a state $|\psi\rangle : \mathbb{C} \to (a)$ and an automorphic recoupling $\mathfrak{i} : a \to a$, the following diagram is commutative.

$$\begin{array}{ccc} (a) & \xrightarrow{(\mathfrak{i})} & (a) \\ \uparrow{\scriptstyle |\psi\rangle} & \nearrow{\scriptstyle \tau_\pi|\psi\rangle} & \\ \mathbb{C} & & \end{array}$$

where π is the permutation of particles given by $\mathfrak{i}$ and $\tau_\pi(a_1 \otimes \cdots \otimes a_n) = a_{\pi 1} \otimes \cdots \otimes a_{\pi n}$. The map $i : a \to a$ represents the recoupling of identical particles by permuting amongst themselves their order in the ensemble a. Alternatively, given any map $|\psi\rangle : \mathbb{C} \to (a)$, a state of the system is given by

$$\sum_{\mathfrak{i}} (\mathfrak{i})|\psi\rangle \tag{6}$$

where we sum over all recouplings $\mathfrak{i} : a \to a$. If all the particle labels of a are distinct then the only recoupling is the identity.

We define an equivalence on the set of ensembles given by $a \sim b$ if and only if there is an ensemble c such that a and b are contained in the direct sum decomposition of c (written $a, b \subset c$). In other words the ensemble c may interact in some way to become either a or b (ignoring dynamical and kinematic considerations). The set of equivalence classes $[a] = \{b : a \sim b\}$ forms an Abelian group $\mathbb{A}$ with addition $[a] + [b] = [a \otimes b]$ and identity $0 = [e]$. The inverse of $[a]$ is given by $-[a] = [a^*]$ since $e \subset a \otimes a^*$. To give some examples, if $G = SU(n)$ ($n \geqslant 2$) then $\mathbb{A} = \mathbb{Z}_n$. If $G = C_n$ and $n \geqslant 1$ then $\mathbb{A} = \mathbb{Z}_{2n}$. If $G = SO(2n+1)$ then $\mathbb{A} = \mathbb{Z}_1$. If $G = D_n$ where $n \geqslant 2$, or G is the tetrahedral, octahedral or icoshedral group then $\mathbb{A} = \mathbb{Z}_2$.

The natural square property of the recouplings mapped under the restriction functor is required to be natural at the state level. This allows us to conclude that the recouplings are of the form

$$\mathfrak{a}_{a,\underset{\cdot}{b},c}((a_i \otimes b_j) \otimes c_k) = \alpha_{m,n,p} a_i \otimes (b_j \otimes c_k) \tag{7}$$

$$\mathfrak{c}_{a,b}(a_i \otimes b_j) = \gamma_{m,n} b_j \otimes a_i \tag{8}$$

$$\mathfrak{l}_a(e \otimes a_i) = \lambda_m a_i \tag{9}$$

$$\mathfrak{r}_a(a_i \otimes e) = \rho_m a_i \tag{10}$$

where $m = [a], n = [b], p = [c]$, $\{a_i\}_i$ is a basis for a, $\{b_j\}_j$ is a basis for b and $\{c_k\}_k$ is a basis for c (see the Appendix for details). All factors $\alpha_{m,n,p}$, $\gamma_{m,n}$, λ_m and ρ_m are phases. The pentagon, hexagon, symmetry and triangle conditions place the following constraints on the phases:

$$\alpha_{m+n,p,q}\alpha_{m,n,p+q} = \alpha_{m,n,p}\alpha_{m,n+p,q}\alpha_{n,p,q} \tag{11}$$

$$\alpha_{m,n,p}\gamma_{m,n+p}\alpha_{n,p,m} = \gamma_{m,n}\alpha_{n,m,p}\gamma_{m,p} \tag{12}$$

$$\gamma_{m,n}\gamma_{n,m} = 1 \tag{13}$$

$$\alpha_{m,0,n}\lambda_n = \rho_m. \tag{14}$$

Any choice of phase factors satisfying these conditions defines a recoupling. We give the following examples:

(i) We have the (pure) Bose recoupling where all phases are unity. If $\mathbb{A} = \mathbb{Z}_1$ the only recoupling is a Bose recoupling.

(ii) If $\mathbb{A} = \mathbb{Z}_2$ the Bose–Fermi recoupling is given by $\gamma_{1,1} = -1$ with all other phases must be unity. Compatibility of states with this recoupling leads to symmetric states for bosons (even grade) and anti-symmetric states for fermions (odd grade). From this follows the Pauli exclusion principle.

(iii) If $\mathbb{A} = \mathbb{Z}_n$ then by the hexagon condition (12) associative recouplings satisfy $\gamma_{m+p,q} = \gamma_{m,q}\gamma_{p,q}$. The general solution is easily found by induction to be $\gamma_{p,q} = (\gamma_{1,1})^{pq}$. The symmetry condition (13) gives $\gamma_{1,1} = \pm 1$. Hence there are only two associative recouplings. Given an $\mathbb{A}$-graded associative algebra $\mathcal{A} = \oplus_{m\in\mathbb{A}} A_m$ one may construct the bracket

$$[a, b] = ab - \gamma_{m,n} ba \tag{15}$$

satisfying $[b, a] = -\gamma_{n,m}[a, b]$ where $a \in \mathcal{A}_m$ and $b \in \mathcal{A}_n$. This bracket satisfies the Jacobi identity $[a, [b, c]] = [[a, b], c] + \gamma_{n,m}[b, [a, c]]$ where $c \in \mathcal{A}_p$. The algebra A with this bracket is a Lie algebra for $\gamma_{1,1} = 1$ and a graded Lie algebra for $\gamma_{1,1} = -1$.

(iv) If we have recouplings $\alpha_{m,n,p}$, $\gamma_{m,n}$, λ_m and ρ_n for $m, n, p \in \mathbb{A}$, and $\alpha'_{m',n',p'}$, $\gamma'_{m',n'}$, $\lambda'_{m'}$ and $\rho'_{n'}$ for $m', n', p' \in \mathbb{A}'$ then the point-wise product $\alpha_{m,n,p}\alpha'_{m',n',p'}$, $\gamma_{m,n}\gamma'_{m',n'}$, $\lambda_m\lambda'_{m'}$ and $\rho_n\rho'_{n'}$ is a recoupling for $\mathbb{A} \times \mathbb{A}'$.

An extension of graded Lie algebras utilizing the recoupling phase algebra here is given in Joyce [26].

In QCD one would like to introduce $SU(3)$ colour and require that it carries a Bose–Fermi recoupling. However, $\mathbb{A} = \mathbb{Z}_3$ obstructs the recoupling from being a symmetric or braided monoidal structure. Let 1 be the class containing the $SU(3)$ representation [3] and 2 its dual $\overline{[3]}$. We require $\gamma_{1,1} = \gamma_{2,2} = -1$. A symmetric monoidal recoupling requires $\gamma_{2,2} = 1$ as we now show. The hexagon condition (12) with $m = n = p = 1$ gives $\alpha_{1,1,1}\gamma_{1,2}\alpha_{1,1,1} = \gamma_{1,1}^2\alpha_{1,1,1}$. Thus the symmetry condition (13) implies $\gamma_{2,1} = \alpha_{1,1,1}$. This together with the hexagon condition (12) with $m = 2$ and $n = p = 1$ gives

$$\gamma_{2,2} = \frac{\alpha_{1,1,1}^2\alpha_{1,2,1}}{\alpha_{2,1,1}\alpha_{1,1,2}}. \qquad (16)$$

But the pentagon condition (11) with $m = n = p = q = 1$ implies that $\gamma_{2,2} = 1$. Hence the colour recoupling cannot be a symmetric monoidal recoupling. Even though such a recoupling may be non-associative, it is too restrictive. Two possibilities exist: a braided monoidal recoupling (see Joyal and Street [13]) and a symmetric premonoidal recoupling (see Joyce [15, 16]). However, the braided monoidal recoupling cannot describe the colour recoupling because the second hexagon equation with $m = n = p = 1$ and the requirement $\gamma_{1,1}^2 = 1$ show $\gamma_{1,2} = \alpha_{1,1,1}$. But from the first hexagon we have $\gamma_{1,2}^{-1} = \alpha_{1,1,1}$. Thus $\gamma_{1,2}^2 = \alpha_{1,1,1}^2 = 1$. Similarly one deduces that $\gamma_{2,1}^2 = \alpha_{2,2,2}^2 = 1$. Importantly, pentagon condition (11) above shows that $\gamma_{2,2} = 1$. There is, however, an important reason why a braid must be symmetric. If we apply commutative recoupling twice to a state $|\psi\rangle : \mathbb{C} \to (a \otimes b)$ we see that $|\psi\rangle = \gamma_{b,a}\gamma_{a,b}|\psi\rangle$ which only admits non-trivial solutions when symmetry condition (13) holds. Only a symmetric premonoidal recoupling is capable of describing a colour recoupling as we demonstrate in the next section.

3. Symmetric premonoidal recoupling

We begin by carefully revisiting the notion of coupling. A coupling tree is a rooted planar binary tree with a linear ordering of its vertices such that every shortest path from the root to a leaf is an increasing sequence and a linear ordering of its leaves. An example is given in figure 1. One should note that the level of the vertices in the tree determines the coupling hierarchy. In this example the coupling sequence is represented by the linear ordering 1324 given by reading the regions from left to right. An ensemble tree is given by evaluation by irrep labels. Given a tuple of labels, we label the leaf in the ith position of the linear ordering by the labelled l_i. The recouplings are represented by unique arrows between coupling trees characterized by a pair of permutations. Note that many coupling trees evaluate to the same ensemble tree. The canonical functor **can** maps ensemble trees to ensembles and recouplings to natural isomorphisms in the obvious way. An example is given in figure 2. The ensemble tree represents physically distinct coupling scenarios that take into account particle indistinguishability. The coupling trees serve to distinguish recouplings and the ensembles are

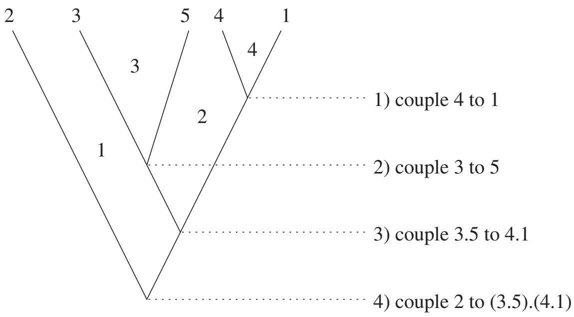


Figure 1. An example of a coupling tree.

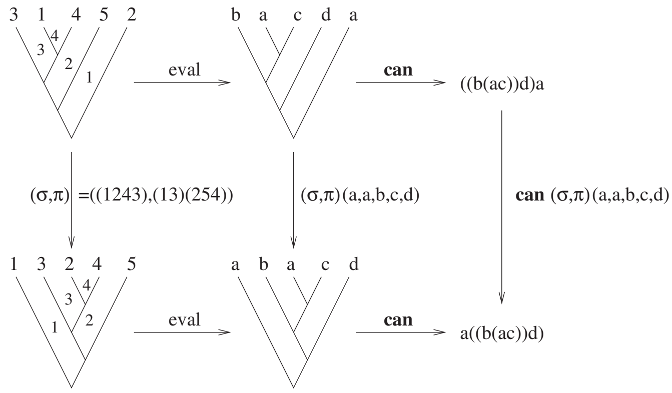


Figure 2. An example of the recoupling σ between two coupling trees, their evaluation by (a, a, b, c, d) to ensemble trees and subsequent mapping under **can** to ensembles.

the state spaces. The permutation σ permutes the coupling sequence, and the permutation π permutes the order of the particles. For a comprehensive exposition see Joyce [16, 17].

We introduce a deformativity natural automorphism $\mathfrak{q}$ to represent the non-commutativity of the pentagon diagram. This is depicted in figure 3. Thus for example, in the ensemble $(a \otimes b) \otimes (c \otimes d)$ we distinguish between coupling a to b before, as opposed to after, coupling c to d. The functor **can** is coherent if the hexagon diagram and triangle diagrams commute, and the following three diagrams commute.

$$(e \otimes a) \otimes b \xrightarrow{e,a,b} e \otimes (a \otimes b), \quad (e \otimes a) \otimes b \xrightarrow{a \otimes 1_b} a \otimes b, \quad e \otimes (a \otimes b) \xrightarrow{a \otimes b} a \otimes b$$

$$(a \otimes b) \otimes e \xrightarrow{a,b,e} a \otimes (b \otimes e), \quad (a \otimes b) \otimes e \xrightarrow{a \otimes b} a \otimes b, \quad a \otimes (b \otimes e) \xrightarrow{1_a \otimes\ b} a \otimes b$$

$$\begin{array}{ccc} (a \otimes b) \otimes (c \otimes d) & \xrightarrow{\mathfrak{q}_{a,b,c,d}} & (a \otimes b) \otimes (c \otimes d) \\ \downarrow {\scriptstyle a\otimes b, c\otimes d} & & \downarrow {\scriptstyle a\otimes b, c\otimes d} \\ (c \otimes d) \otimes (a \otimes b) & \xleftarrow{\mathfrak{q}_{c,d,a,b}} & (c \otimes d) \otimes (a \otimes b). \end{array}$$

The deformativity recoupling is given by (see the appendix)

$$\mathfrak{q}((a_i \otimes b_j) \otimes (c_k \otimes d_l)) = \xi_{a,b,c,d}(a_i \otimes b_j) \otimes (c_k \otimes d_l) \tag{17}$$

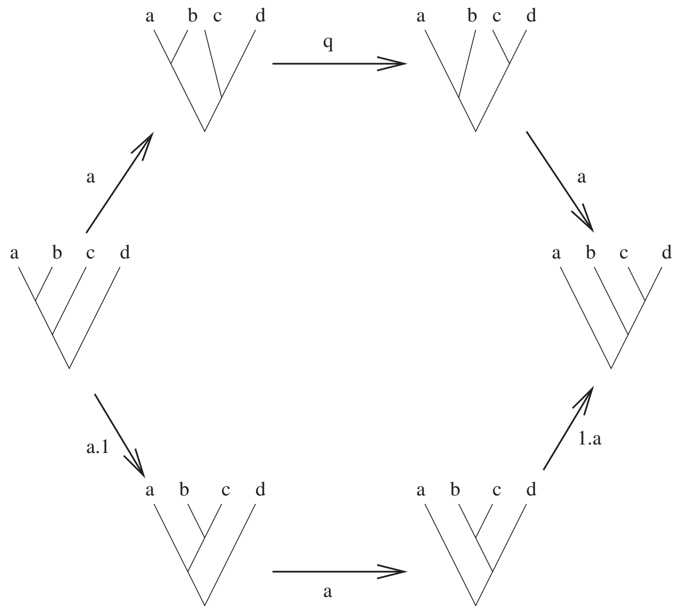


Figure 3. The q-pentagon diagram of a premonoidal structure, where q represents the degree to which the pentagon diagram does not commute.

where $\xi_{a,b,c,d}$ is a phase factor and a class function of the $\mathbb{A}$-gradation. The constraints on the recoupling phases are

$$\alpha_{m+n,p,q}\xi_{m,n,p,q}\alpha_{m,n,p+q} = \alpha_{m,n,p}\alpha_{m,n+p,q}\alpha_{n,p,q} \tag{18}$$

$$\alpha_{m,n,p}\gamma_{m,n+p}\alpha_{n,p,m} = \gamma_{m,n}\alpha_{n,m,p}\gamma_{m,p} \tag{19}$$

$$\xi_{m,n,p,q}\xi_{p,q,m,n} = 1 \tag{20}$$

$$\gamma_{m,n}\gamma_{n,m} = 1 \tag{21}$$

$$\alpha_{0,m,n}\lambda_{m+n} = \lambda_m \tag{22}$$

$$\alpha_{m,0,n}\lambda_n = \rho_m \tag{23}$$

$$\alpha_{m,n,0}\rho_n = \rho_{m+n} \tag{24}$$

for all $m, n, p, q \in \mathbb{A}$. Note that (18) provides a formula for $\xi_{m,n,p,q}$. Let $S^1 = \{z \in \mathbb{C} : |z| = 1\} \subset \mathbb{C}$ be the set of phase factors. We now give a formal definition of a recoupling for an Abelian group $\mathbb{A}$.

Definition 1. *A recoupling for an Abelian group $\mathbb{A}$ consists of the four maps $\alpha : \mathbb{A}^3 \to S^1$, $\gamma : \mathbb{A}^2 \to S^1$ and $\lambda, \rho : \mathbb{A} \to S^1$ satisfying conditions (18)–(24).*

A recoupling is called a Bose–Fermi recoupling whenever $\gamma_{m,m} = -1$ for all $m \in \mathbb{A} \setminus \{0\}$. We can define a Bose–Fermi recoupling for any $\mathbb{A}$-gradation as follows. We take $\lambda_m = \rho_n = 1$ and

$$\gamma_{m,n} = \begin{cases} 1 & m = 0 \text{ or } n = 0 \\ -1 & \text{otherwise} \end{cases} \tag{25}$$

$$\alpha_{m,n,p} = \begin{cases} 1 & m = 0, n = 0, p = 0 \text{ or } m + n = 0 \\ -1 & \text{otherwise.} \end{cases} \tag{26}$$

The $m + n = 0$ in the definition of $\alpha_{m,n,p}$ may equally well be replaced by $n + p = 0$. These determine the deformativity phases to be

$$\xi_{m,n,p,q} = \begin{cases} 1 & m = 0, n = 0, p = 0, q = 0, m + n = 0 \text{ or } p + q = 0 \\ -1 & \text{otherwise.} \end{cases} \tag{27}$$

We immediately see that the recoupling is monoidal for $\mathbb{A} = \mathbb{Z}_2$, but premonoidal for $\mathbb{A} = \mathbb{Z}_n$ where $n \geqslant 3$. To verify the phase conditions we only need to demonstrate that hexagon condition (19) holds and that the definition of $\xi_{m,n,p,q}$ is correct, the other conditions are immediate. If $m = 0$, $n = 0$ or $p = 0$, this is easily shown. Suppose they are all non-zero, then $\gamma_{m,n}\gamma_{m,p} = 1$. If $n + p = 0$ then the hexagon condition reduces to $\alpha_{m,n,-n}\alpha_{n,-n,m} = \alpha_{n,m,-n}$ which holds. Now also suppose that $n + p \neq 0$ then $\gamma_{m,n+p} = -1$ and the hexagon condition is $\alpha_{m,n,p}\alpha_{n,p,m} = -\alpha_{n,m,p}$ which holds. A similar argument shows the definition of $\xi_{m,n,p,q}$ is correct.

For this $\mathbb{Z}_3$-graded Bose–Fermi recoupling all phases are unity except the following which are -1:

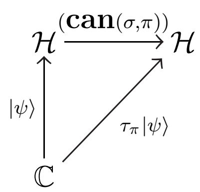
$$\begin{array}{lll} \gamma_{1,1} & \alpha_{1,1,1} & \xi_{1,1,1,1} \\ \gamma_{1,2} & \alpha_{1,1,2} & \xi_{1,1,2,2} \\ \gamma_{2,1} & \alpha_{2,2,1} & \xi_{2,2,1,1} \\ \gamma_{2,2} & \alpha_{2,2,2} & \xi_{2,2,2,2}. \end{array} \tag{28}$$

4. Exclusion and confinement principles

Given an ensemble of particles, sometimes there are a number of coupling schemes associated with it. This occurs when there are identical particles, or when the coupling process is non-monoidal. These situations lead respectively, to exclusion and confinement principles.

Indistinguishability requirements place statistical constraints on what states of a given system are possible. Given an ensemble tree w, the state space of the system is $\mathcal{H} = (\mathbf{can}\, w)$. Thus a map $|\psi\rangle : \mathbb{C} \to \mathcal{H}$ is a state of the system if it is compatible with the two following conditions.

(i) *Indistinguishability of particles*: given ensemble trees w and w' with the same state space $\mathcal{H}$, $|\psi\rangle : \mathbb{C} \to \mathcal{H}$ is a state of the system if for every recoupling $(\sigma, \pi) : w \to w'$ the diagram below commutes.



where $\tau_\pi(a_1 \otimes \cdots \otimes a_n) = a_{\pi 1} \otimes \cdots \otimes a_{\pi n}$.

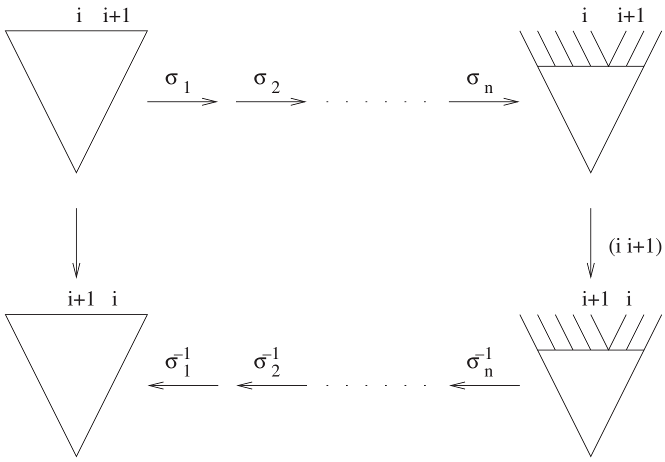
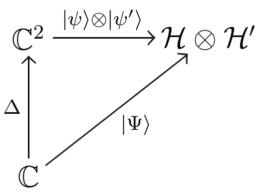


Figure 4. Transposition of two adjacent particles.

(ii) *Composition of particles*: given two states $|\psi'\rangle : \mathbb{C}\mathcal{H} \to \mathcal{H}'$ and $|\psi'\rangle : \mathbb{C} \to \mathcal{H}$ the composite $|\Psi\rangle : \mathbb{C} \to \mathcal{H} \otimes \mathcal{H}'$ given by the commuting of the diagram below is a state (and so satisfies (i)).



where $\Delta z = (z, z)$ for all $z \in \mathbb{C}$ is the diagonal map.

Note that if the recoupling is symmetric monoidal then property (ii) follows from (i).

The next result deduces the generalization of the Pauli exclusion principle.

Principle 1 (Exclusion)**.** *Given an ensemble of identical particles a, the Bose–Fermi recoupling asserts that the state is symmetric if* $a \in 0$ *and anti-symmetric otherwise.*

This justifies the name of the recoupling and is the Pauli exclusion principle for $G = SU(2)$.

Proof. Given any coupling tree w we wish to determine a sequence of associativity and one commutativity recouplings, the interchange of the ith and $(i + 1)$th leaf. To do this determine a sequence of associativity recouplings that ensures the ith and $(i + 1)$th leafs are coupled together first in the coupling tree. Next apply the commutativity recoupling swapping them, and finally reverse the sequence of associativity recouplings to give a coupling tree w' that only differs from w by the interchange of the ith and $(i + 1)$th leaves. This is depicted in figure 4. Next evaluate these trees for a fixed label a. They give rise to the same ensemble tree, and under **can** the same ensemble. The recoupling phase is given by $\gamma_{a,a}$ since all the associativity recoupling phases must cancel by construction. Thus any state under adjacent interchange introduces a phase factor $\gamma_{a,a}$. Hence by indistinguishability a state of the system is symmetric if $\gamma_{a,a} = 1$ and anti-symmetric for $\gamma_{a,a} = -1$. □

We now deduce the principle of state confinement.

Principle 2 (Confinement)**.** *Given a Bose–Fermi recoupling, there is a nilpotent n of $\mathbb{A}$ (that is $2n = 0$) such that the non-zero states correspond to ensembles of grade zero and n.*

If $\mathbb{A}$ has no non-zero nilpotent grades, the non-zero states are confined to grade zero ensembles. This is the situation for $SU(3)$ colour giving quark state confinement.

Proof. We begin by proving that

$$\xi_{m,n,m,n} = \gamma_{m+n,m+n}\gamma_{m,m}\gamma_{n,n}. \tag{29}$$

Hexagon condition (19) gives $\alpha_{m+n,m,n}\alpha_{m,n,m+n} = \alpha_{m,m+n,n}\gamma_{m+n,m+n}\gamma_{m+n,m}\gamma_{m+n,n}$. Substituting this into formula (18) for $\xi_{m,n,m,n}$ gives $\xi_{m,n,m,n} = \alpha_{m,n,m}\alpha_{n,m,n}\gamma_{m+m,m+n}\ \gamma_{m,m+n}\gamma_{n,m+n}$. Again hexagon condition (19) gives $\alpha_{m,n,m} = \gamma_{m,n}\gamma_{m,m}\gamma_{m+n,m}$, and a similar formula with m and n interchanged. Substituting these into the previous expression gives the desired formula. If a corresponds to an ensemble for which its grade $[a] = m$ does not generate $\mathbb{Z}_1$ or $\mathbb{Z}_2$ then $\xi_{m,m,m,m} = \gamma_{2m,2m} = -1$. Now the composition of state property applied to a state $|\psi\rangle : \mathbb{C} \to (a)$ gives the 4-fold composite state $|\Psi\rangle : \mathbb{C} \to ((a \otimes b) \otimes (a \otimes b))$ satisfying $|\psi\rangle = \xi_{m,n,m,n}|\psi\rangle$. This can only occur if $|\psi\rangle = 0$. The ensembles admitting non-trivial states generate an Abelian subgroup $\mathbb{A}_0$ of grades m, n satisfying $m + n = 0$ because if $m + n \neq 0$ either m or n would admit only trivial states. Hence $\mathbb{A}_0$ is $\mathbb{Z}_1$ or $\mathbb{Z}_2$ giving the desired nilpotent. Either way the deformativity phase is always zero. □

For $SU(2)$, which is $\mathbb{Z}_2$-graded, one arrives at the conclusion that the only non-unity phase possible is $\gamma_{1,1}$. Moreover, the recouplings are symmetric monoidal and there is only one choice of Bose–Fermi recoupling ($\gamma_{1,1} = -1$). Thus Pauli exclusion follows and there is no state confinement requirement. On the other hand for $SU(3)$, which is $\mathbb{Z}_3$-graded, there are a number of Bose–Fermi recouplings. Importantly, they are all symmetric premonoidal (never monoidal), satisfy Pauli exclusion and because of state confinement only triality zero states are possible.

The only remaining $\mathbb{Z}_n$-grade admitting the state confinement observed in nature is $\mathbb{Z}_6$. This could be aligned with $SU(6)$ flavour. However, since each quark flavour has a different mass there is no reason to believe that a flavour indistinguishability principle exists. Moreover, $SU(2)$ spin and $SU(3)$ colour are sufficient to describe the statistical behaviour observed in nature.

In standard QFT the associativity is strict and brackets are ignored. In other words all $\alpha_{m,n,p}$ are unity. In the case of QCD some modification of the recoupling structure is required if confinement is to become an intrinsic property. The only irreducible physical ensembles are the vacuum, mesons, hadrons and free gluons. Gluons are free to enter and exit mesons and hadrons providing the mechanism of the strong interaction. It is important to realise that one cannot have the Pauli exclusion principle for $SU(3)$ colour without the confinement of quarks to mesonic and hadronic ensembles. A formulation of many-body quantum theory taking this into account is given in Joyce [21]. This approach does not rely on annihilation and creation operators. It is an open question as to what form non-associative algebras of annihilation and creation operators might take to accommodate non-associative recoupling. A promising candidate is the extension of graded Lie algebras given in Joyce [26].

5. Conclusion

Starting from fundamental principles we derived the recoupling structure of ensemble quantum systems with exact symmetry. This was found to lead to a recoupling algebra of phases. The symmetry of the situation leads to a gradation for the ensembles of which the recoupling is a class function. There is some freedom in the choice of phases, each leading to different statistical behaviour.

Physical requirements demand the usual Bose–Fermi recoupling over $SU(2)$ spin and $SU(3)$ colour. In order to accommodate this for $SU(3)$ colour we deduced the need for non-associative recoupling. More generally we constructed a consistent Bose–Fermi recoupling for any gradation. The recoupling algebra placed constraints on what states of the system are allowable. For Bose–Fermi recoupling we demonstrated that a (generalized) Pauli exclusion principle holds. Additionally we proved that a state confinement principle was unavoidable. The triality grading of $SU(3)$ colour ensembles ensured that quark state confinement was mandatory. No *confining force* was necessary to explain quark state confinement. However, spatial confinement of quarks to within baryons is explained by the dynamics of a theory such as QCD.

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Appendix

The natural square property of the recouplings mapped under the restriction functor is required to be natural at the state level. Considering commutativity this natural condition is as follows: given $a \cong c$ and $b \cong d$ then

$$\begin{array}{ccc} (a) \otimes (b) & \xrightarrow{(\ \ _{a,b})} & (b) \otimes (a) \\ {\scriptstyle X\otimes Y}\downarrow & & \downarrow{\scriptstyle Y\otimes X} \\ (c) \otimes (d) & \xrightarrow{(\ \ _{c,d})} & (d) \otimes (c) \end{array}$$

commutes for all linear transformations $X : (a) \to (c)$ and $Y : (b) \to (d)$. Suppose

$$(\mathfrak{c}_{a,b}) a_i \otimes b_j = (C_{a,b})_{ij}^{kl} b_l \otimes a_k \tag{30}$$

where $\{a_i\}_i$ is a basis for a and $\{b_j\}_j$ is a basis for b. Take $c = a$, $d = b$, $X = X(i;k) : a_r \mapsto a_k \delta_{i,r}$ and $Y = Y(j;l) : b_s \mapsto b_l \delta_{j,s}$ in the square diagram and apply the maps to the basis vector $a_i \otimes b_j$. The top right half gives

$$a_i \otimes b_j \mapsto \sum_{m,n} (C_{a,b})_{ij}^{mn} b_n \otimes a_m \mapsto (C_{a,b})_{ij}^{ij} b_l \otimes a_k. \tag{31}$$

And the bottom left half gives

$$a_i \otimes b_j \mapsto a_k \otimes b_l \mapsto \sum_{r,s} (C_{a,b})_{kl}^{rs} b_s \otimes a_r. \tag{32}$$

These two being equal allows us to conclude that $(C_{a,b})_{kl}^{rs} = \delta_k^r \delta_l^s (C_{a,b})_{ij}^{ij}$ and hence $\mathfrak{c}_{a,b}$ can only introduce a global phase factor $(C_{a,b})_{11}^{11}$. Moreover, if $a \cong c$ and $b \cong d$ then $\mathfrak{c}_{a,b}$ and $\mathfrak{c}_{c,d}$ introduce the same phase which we denote by $\gamma_{[a],[b]}$. That is to say the commutativity phase is $\mathbb{A}$-graded. Similar arguments allow us to conclude that all recouplings contribute only phase factors.

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